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DIFFUSION THROUGH COMPOSITE MEDIA

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Abstract—The diffusion equation for the temperature distribution in each of k sections of a composite with internal heat generation and either heating, cooling or perfect thermal contact at the $k-1$ interfaces is solved. The composite consists of k discrete plates, cylinders or spheres each of different material. The k sections have an arbitrary initial temperature distribution and the media are exchanging heat at the external boundaries through two different, though constant, arbitrary film coefficients with two different time dependent surroundings.

The solution is obtained by using a unique dependent variable substitution which gives a new partial differential equation with homogeneous external boundary conditions. The solution for this derived partial differential equation is then obtained by using the Vodicka type of orthogonality relationship.

The solution, thus obtained, gives the temperature distribution in any of the k plates, cylinders or spheres for any position x and time t for the most general type of linear boundary, internal and initial conditions.

NOMENCLATURE

$\|a\|$	determinant;
A_{ij}, B_{ij},	constants;
A_{im}, B_{im},	constants;
$a_{\xi\eta}$,	elements of a determinant;
$\|b\|$,	determinant;
c,	constant;
c_{pi},	specific heat at constant pressure in the ith section;
D, D_m,	determinant;
D_i^2,	thermal diffusivity in the ith section;
$F_j(t)$,	function of time, defined by equations (8) and (9);
$G_i(x)$,	derived function of x, defined by equation (10);
g_m,	constant, defined by equation (43);
h_1, h_{k+1},	surface film coefficients at the external boundaries;
h_{i+1},	surface film coefficient at $x = x_{i+1}$, $i = 1, 2, 3, \ldots, k-1$;
i, j, k,	integers;
K_i	thermal conductivity in the ith section;
$L_{ij}(x)$,	function of x, defined by equation (4);
l_{mj},	constant, defined by equation (41);
M_{im}, N_{im},	derived eigenfunction terms [dimensionless];
m, n,	integers;
N_m,	sum of weighted integrals;
P,	constant;
Q_i,	distributed source in the ith section;
$q_m(t)$,	function of time, defined by equation (42);
t,	time;
$T_i(x, t)$,	temperature in the ith section;
$U_i(x, t)$,	derived temperature in the ith section;
$u_m(t)$,	function of time, defined by equation (27);
$V_i(x)$,	initial condition in the ith section;
v_m,	constant, defined by equation (49);
W_i,	weighting function;
$X_{im}(x)$,	eigenfunction [dimensionless];
x,	spatial coordinate;
$y_1(t), y_2(t)$,	temperature of surroundings;
z,	term in matrix equation;
∇^2,	Laplacian operator;
β, ξ, η,	integers;
γ_n,	eigenvalue;

$\theta(x)$, defined by equation (6);
λ_1, ratio of thermal conductivities, dimensionless;
ρ_i, density in the ith section.

INTRODUCTION

THE TREATMENT of problems in diffusion through composite media have been handled by many authors; some examples of these methods are given in *Conduction of Heat in Solids* by H. S. Carslaw and J. C. Jaeger in which they used complex variable methods and residue theory to obtain the temperature distribution. The difficulty with the above mentioned techniques is that for composites with more than two layers the usual analytical techniques become very formidable and practically preclude obtaining solutions in this manner. The first significant contribution to solving problems of this type occurred in two classic papers by V. Vodicka [8, 9] in 1950 and 1955. The method was independently developed by C. W. Tittle [7] in 1965. These solutions are based on a new type of orthogonality relationship in which the orthogonal sets are constructed from the non-orthogonal eigenset, which comes about from the separation of variables method of solution, by using orthogonality factors for each layer which are obtained from the orthogonality condition. This method was used by Bulavin and Kascheev [2] in 1965 to solve the heat conduction problem with heat generation in a composite domain of k sections for plates, cylinders or spheres that are insulated on one boundary and are exchanging heat with a time dependent surrounding on the other external boundary. In 1967, Beach [1] treated conduction heat transfer in multilayered cylinders with temperature continuity at the interfaces and Moore [5] solved the problem for a two-layer slab with a temperature discontinuity at the interface; both author's used simple external boundary conditions to obtain their solutions. A fairly complete review of the literature in the analysis of composite media is given by Ozisik [6].

In this paper, the particular problem of heat conduction through composite media is solved; the problem that is considered is of the most general type for plates, cylinders and spheres. The composite media have an arbitrary initial temperature distribution, and are subject to being heated or cooled at their mutual external boundaries by two different, arbitrary, time dependent surroundings, each having a different arbitrary constant film coefficient through which heat is transferred. At the $k - 1$ interfacial boundaries, the media are subject to being heated or cooled through different arbitrary constant film coefficients through which heat is transferred along with the condition of continuity of heat flux: for very large values of the film coefficient ($h_\xi \to \infty, \xi = 2, 3, 4, \ldots, k$) at the interfacial boundaries, the problem reduces to that of temperature continuity at the interface. The solution is developed by using a unique dependent variable transformation which gives a new partial differential equation with homogeneous external boundary conditions. The solution for this new partial differential equation is obtained by using the Vodicka type of orthogonality relationship.

When one external film coefficient is zero and all the interfacial film coefficients are very large, the method gives solutions to the Bulavin and Kascheev type of problem. Additionally, the method reduces to a standard Sturm–Liouville problem when $k = 1$, giving classical solutions for problems of a single component.

PROBLEM

The temperature distribution in the ith section of k solidly joined plates, cylinders or spheres is given by the diffusion equation:

$$\nabla^2 T_i(x, t) + \frac{Q_i(x, t)}{K_i} = \frac{1}{D_i^2}\frac{\partial T_i(x, t)}{\partial t} \tag{1}$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, 3, \ldots, k, t \geqslant 0$$

where

∇^2 = Laplacian operator
T_i = temperature in the ith section

Q_i = distributed source in the ith section
K_i = thermal conductivity in the ith section
$D_i^2 = K_i/\rho_i C_{pi}$ = thermal diffusivity in the ith section
ρ_i = density in the ith section
C_{pi} = specific heat in the ith section.

The composite media are subject to the following external boundary and initial conditions:

(a) $$K_1 \frac{\partial T_1(x_1, t)}{\partial x} = h_1[T_1(x_1, t) - y_1(t)]$$

(b) $$K_k \frac{\partial T_k(x_{k+1}, t)}{\partial x} = -h_{k+1}[T_k(x_{k+1}, t) - y_2(t)] \qquad (2)$$

(c) $$T_i(x, 0) = V_i(x), x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, \ldots, k$$

where

h_1, h_{k+1} = external surface film coefficients $0 \leqslant h_1,\ h_{k+1} \leqslant \infty$ (one but not both may be zero)
$y_1(t), y_2(t)$ = temperature of surroundings.

The problem where both surface film coefficients are zero corresponds to either insulated external boundaries or a known heat flux input at the external boundaries and the solutions can be obtained in the same manner as given here.

At the internal boundaries, the media are subject to the following conditions:

(a) $$-K_i \frac{\partial T_i(x_{i+1}, t)}{\partial x} = h_{i+1}[T_i(x_{i+1}, t) - T_{i+1}(x_{i+1}, t)] \qquad (3)$$

(b) $$K_i \frac{\partial T_i(x_{i+1}, t)}{\partial x} = K_{i+1} \frac{\partial T_{i+1}(x_{i+1}, t)}{\partial x}$$

where

h_{i+1} = surface film coefficient at $x = x_{i+1}$, $i = 1, 2, \ldots, k$

To obtain homogeneous external boundary conditions, let

$$T_i(x, t) = U_i(x, t) + \sum_{j=1}^{2} L_{ij}(x) F_j(t) \qquad (4)$$
$$x_i \leqslant x \leqslant x_{i+1},\ i = 1, 2, \ldots, k, t \geqslant 0$$

and further let

$$\nabla^2 L_{ij}(x) = 0 \qquad (5)$$

so that

$$L_{ij}(x) = A_{ij}\theta(x) + B_{ij} \qquad (6)$$
$$x_i \leqslant x \leqslant x_{i+1},\ i = 1, 2, 3, \ldots, k,\ j = 1, 2.$$

The function $\theta(x)$ is given in Table 1 for rectangular, cylindrical and spherical geometries. Substitution of equations (4) and (5) into equation (1) gives

$$D_i^2 \nabla^2 U_i(x, t) + D_i^2 \frac{Q_i(x, t)}{K_i} = \frac{\partial U_i(x, t)}{\partial t} + \sum_{j=1}^{2} L_{ij}(x) \frac{dF_j(t)}{dt} \qquad (7)$$
$$x_i \leqslant x \leqslant x_{i+1},\ i = 1, 2, 3, \ldots, k, t \geqslant 0$$

where

$$F_1(t) = -y_1(t) \qquad (8)$$

and

$$F_2(t) = y_2(t) \qquad (9)$$

Table 1.

Geometry	$\theta(x)$
Plate	x
Cylinder	$\ln x$
Sphere	$1/x$

For $U_i(x, t)$, the initial conditions are

$$U_i(x, 0) = V_i(x) - \sum_{j=1}^{2} L_{ij}(x) F_j(0) = G_i(x) \qquad (10)$$

the external boundary conditions are

$$\begin{aligned}
&\text{(a)}\quad K_1 \frac{\partial U_1(x_1, t)}{\partial x} - h_1 U_1(x_1, t) = 0 \\
&\text{(b)}\quad K_k \frac{\partial U_k(x_{k+1}, t)}{\partial x} + h_{k+1}\, U_k(x_{k+1}, t) = 0
\end{aligned} \tag{11}$$

and the internal boundary conditions are

$$\begin{aligned}
&\text{(a)}\quad -K_i \frac{\partial U_i}{\partial x}(x_{i+1}, t) = h_{i+1}\,[U_i(x_{i+1}, t) \\
&\qquad\qquad - U_{i+1}(x_{i+1}, t)] \\
&\text{(b)}\quad K_i \frac{\partial U_i(x_{i+1}, t)}{\partial x} = K_{i+1} \frac{\partial U_{i+1}(x_{i+1}, t)}{\partial x}.
\end{aligned} \tag{12}$$

The external boundary conditions for $L_{i1}(x)$ are

$$\begin{aligned}
&\text{(a)}\quad K_1 \frac{\mathrm{d}L_{11}(x_1)}{\mathrm{d}x} - h_1 L_{11}(x_1) = h_1 \\
&\text{(b)}\quad K_k \frac{\mathrm{d}L_{k1}}{\mathrm{d}x}(x_{i+1}) + h_{k+1} L_{k1}(x_{k+1}) = 0
\end{aligned} \tag{13}$$

and the internal boundary conditions for $L_{i1}(x)$ are

$$\begin{aligned}
&\text{(a)}\quad -K_i \frac{\mathrm{d}L_{i1}(x_{i+1})}{\mathrm{d}x} = h_{i+1}[L_{i1}(x_{i+1}) \\
&\qquad\qquad - L_{i+1,1}(x_{i+1})] \\
&\text{(b)}\quad K_i \frac{\mathrm{d}L_{i1}(x_{i+1})}{\mathrm{d}x} = K_{i+1} \frac{\mathrm{d}L_{i+1,1}(x_{i+1})}{\mathrm{d}x}
\end{aligned} \tag{14}$$

For $L_{i2}(x)$, the external boundary conditions are

$$\begin{aligned}
&\text{(a)}\quad K_1 \frac{\mathrm{d}L_{12}(x_1)}{\mathrm{d}x} - h_1 L_{12}(x_1) = 0 \\
&\text{(b)}\quad K_k \frac{\mathrm{d}L_{k2}(x_{k+1})}{\mathrm{d}x} + h_{k+1} L_{k2}(x_{k+1}) = h_{k+1}
\end{aligned} \tag{15}$$

and the internal boundary conditions are

$$\begin{aligned}
&\text{(a)}\quad -K_i \frac{\mathrm{d}L_{i2}(x_{i+1})}{\mathrm{d}x} = h_{i+1}[L_{i2}(x_{i+1}) \\
&\qquad\qquad - L_{i+1,2}(x_{i+1})] \\
&\text{(b)}\quad K_i \frac{\mathrm{d}L_{i2}(x_{i+1})}{\mathrm{d}x} = K_{i+1} \frac{\mathrm{d}L_{i+1,2}(x_{i+1})}{\mathrm{d}x}
\end{aligned} \tag{16}$$

$$\left[\begin{array}{ccccccc}
K_1\theta'(x_1) - h_1\theta(x_1) & -h_1 & 0 & 0 & 0 & . & . \\
\frac{K_1}{h_2}\theta'(x_2) + \theta(x_2) & 1 & -\theta(x_2) & -1 & 0 & . & . \\
\lambda_1 & 0 & -1 & 0 & 0 & . & . \\
0 & 0 & \frac{K_2}{h_3}\theta'(x_3) + \theta(x_3) & 1 & -\theta(x_3) & . & . \\
0 & 0 & \lambda_2 & 0 & -1 & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & . & . \\
0 & 0 & 0 & 0 & 0 & . & . \\
0 & 0 & 0 & 0 & 0 & . & .
\end{array}\right.$$

Solution, $L_{i1}(x)$

The solution for $L_{i1}(x)$ is obtained by applying the internal and external boundary conditions, equations (13) and (14), to equation (6); the result is a system of $2k$ linear non-homogeneous equations for determining the $2k$ constants A_{i1} and B_{i1}.

In matrix form, the $2k$ simultaneous equations are given by equation (17) and in matrix notation they can be represented by

$$[a]\{z\} = \{P\} \tag{18}$$

where the elements of the square matrix $a_{\xi\eta}$ $\xi, \eta, = 1, 2, 3, \ldots, 2k$ are obvious and where

$$\begin{aligned} z_1 &= A_{11}, \quad z_2 = B_{11}, \\ z_3 &= A_{21}, \quad z_4 = B_{21}, \\ z_{2\beta-1} &= A_{\beta 1}, \quad z_{2\beta} = B_{\beta 1}, \quad \beta = 1, 2, 3, \ldots, k \\ z_{2k-1} &= A_{k1}. \quad z_{2k} = B_{k1} \end{aligned} \tag{19}$$

and

$$\begin{aligned} P_1 &= h_1 \\ P_\beta &= 0, \quad \beta = 2, 3, 4, \ldots, 2k. \end{aligned} \tag{20}$$

Equation (17) has a unique solution given by

$$\begin{aligned} z_1 &= \frac{|D_1|}{|a|}, \quad z_2 = \frac{|D_2|}{|a|}, \ldots, \\ z_\beta &= \frac{|D_\beta|}{|a|}, \ldots, \quad z_{2k} = \frac{|D_{2k}|}{|a|} \end{aligned} \tag{21}$$

where D_β is the determinant formed by replacing the elements of the βth column $(a_{1\beta}, a_{2\beta}, \ldots, a_{2k,\beta})$ by the P column $(P_1, P_2, \ldots, P_{2k})$ in the determinant of the coefficient matrix, $|a|$. Thus all the A_{i1} and B_{i1} can be determined.

Solution $L_{i2}(x)$

The solution for $L_{i2}(x)$ is obtained in the same manner as $L_{i1}(x)$; the only exception is that in this case

$$\begin{aligned} P_\beta &= 0, \quad \beta = 1, 2, 3, \ldots, 2k-1 \\ P_{2k} &= h_{k+1} \end{aligned} \tag{22}$$

$$\begin{array}{ccccc} & 0 & 0 & & 0 \\ & 0 & 0 & & 0 \\ & 0 & 0 & & 0 \\ & 0 & 0 & & 0 \\ & 0 & 0 & & 0 \\ & \cdot & \cdot & & \cdot \\ & \cdot & \cdot & & \cdot \\ & \cdot & \cdot & & \cdot \\ \frac{k-1}{h_k}\theta'(x_k) + \theta(x_k) & 1 & -\theta(x_k) & & -1 \\ k-1 & 0 & -1 & & 0 \\ & 0 & K_k\theta'(x_{k+1}) + h_{k+1}\theta(x_{k+1}) & & h_{k+1} \end{array} \left[\begin{array}{c} {}_{11} \\ B_{11} \\ A_{21} \\ B_{21} \\ A_{31} \\ \cdot \\ \cdot \\ \cdot \\ B_{k-1,1} \\ A_{k1} \\ B_{k1} \end{array}\right] = \left[\begin{array}{c} h_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{17}$$

and all the A_{i2} and B_{i2} can be determined.

Solution, $U_i(x, t)$

With the constants A_{i1}, B_{i1}, A_{i2} and B_{i2} known, the problem is sufficiently simplified so that a series solution [2] for equation (7) can be assumed wherein the unique orthogonality conditions developed by Vodicka [7, 8] can be used. Thus

$$U_i(x, t) = \sum_{m=1}^{\infty} u_{im}(t)\, X_{im}(x) \tag{23}$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, 3, \dots, k, \quad t \geqslant 0.$$

When an internal boundary condition such as equation (12b) is used in equation (23), the following result is obtained:

$$K_i \frac{\partial U_i(x_{i+1}, t)}{\partial x} = K_i \sum_{m=1}^{\infty} u_{im}(t) \frac{\mathrm{d}X_{im}(x_{i+1})}{\mathrm{d}x} = K_{i+1} \frac{\partial U_{i+1}(x_{i+1}, t)}{\partial x} = K_{i+1} \sum_{m=1}^{\infty} \times u_{i+1,m}(t) \frac{\mathrm{d}X_{i+1,m}(x_{i+1})}{\mathrm{d}x}. \tag{24}$$

Equation (24) is still an identity if for any m

$$K_i u_{im}(t) \frac{\mathrm{d}X_{im}(x_{i+1})}{\mathrm{d}x} = K_{i+1} u_{i+1,m}(t) \frac{\mathrm{d}X_{i+1,m}(x_{i+1})}{\mathrm{d}x}. \tag{25}$$

For arbitrary t and fixed x_{i+1}, equation (25) can be an identity only if

$$u_{im}(t) = u_{i+1,m}(t) = u_m(t). \tag{26}$$

Thus equation (23) can be replaced by

$$U_i(x, t) = \sum_{m=1}^{\infty} u_m(t) X_{im}(x) \tag{27}$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, 3, \dots, k, t \geqslant 0$$

where $u_m(t)$ is a function to be determined from the initial conditions and $X_{im}(x)$, $x_i \leqslant x \leqslant x_{i+1}$, $i = 1, 2, 3, \dots, k$, are eigenfunctions of the eigenvalue problem

$$\frac{D_i^2}{x^c} \frac{\mathrm{d}}{\mathrm{d}x}\left[x^c \frac{\mathrm{d}X_{im}}{\mathrm{d}x}\right] + \gamma_m^2 X_{im}(x) = 0 \tag{28}$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, 3, \dots, k$$

where the constant c has the value 0 for plates, 1 for cylinders and 2 for spheres and where γ_m is an eigenvalue of the problem. Equation (28) has the following external boundary conditions:

$$\text{(a)} \quad \frac{\mathrm{d}X_{1m}(x_1)}{\mathrm{d}x} - \frac{h_1}{K_1} X_{1m}(x_1) = 0$$

$$\text{(b)} \quad \frac{\mathrm{d}X_{km}(x_{k+1})}{\mathrm{d}x} + \frac{h_{k+1}}{K_k} X_{km}(x_{k+1}) = 0 \tag{29}$$

and the following internal boundary conditions:

$$\text{(a)} \quad \frac{-K_i}{h_{i+1}} \frac{\mathrm{d}X_{im}(x_{i+1})}{\mathrm{d}x} = X_{im}(x_{i+1}) - X_{i+1,m}(x_{i+1})$$

$$\text{(b)} \quad K_i \frac{\mathrm{d}X_{im}(x_{i+1})}{\mathrm{d}x} = K_{i+1} \frac{\mathrm{d}X_{i+1,m}(x_{i+1})}{\mathrm{d}x}. \tag{30}$$

The solution of equation (28) is

$$X_{im}(x) = A_{im} M_{im}(x) + B_{im} N_{im}(x) \tag{31}$$

where $M_{im}(x)$ and $N_{im}(x)$ are linearly independent solutions [2] and are given in Table 2.

Table 2.

Geometry	$M_{im}(x)$	$N_{im}(x)$	m
	1	x	0
Plate ($c = 0$)	$\cos\left(\frac{\gamma_m}{D_i} x\right)$	$\sin\left(\frac{\gamma_m}{D_i} x\right)$	1, 2, 3, ...
	1	$\ln x$	0
Cylinder ($c = 1$)	$J_0\left(\frac{\gamma_m}{D_i} x\right)$	$Y_0\left(\frac{\gamma_m}{D_i} x\right)$	1, 2, 3, ...
	1	$1/x$	0
Sphere ($c = 2$)	$\frac{1}{x} \sin\left(\frac{\gamma_m}{D_i} x\right)$	$\frac{1}{x} \cos\left(\frac{\gamma_m}{D_i} x\right)$	1, 2, 3, ...

The eigenvalues γ_m, $m = 1, 2, 3, \dots$, are obtained by inserting equation (31) into the boundary conditions, equations (29) and (30); the result is a system of $2k$ linear homogeneous equations from which the constants A_{im} and B_{im}, $m = 1, 2, 3, \ldots$, can be determined. This system of equations has a non-trivial solution when its determinant is equal to zero. Setting this determinant equal to zero we obtain the transcendental equation

$$D = D_m = \begin{vmatrix} M^*_{im}(x_1) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & N^*_{im}(x_{k+1}) \end{vmatrix} = 0, m = 1, 2, 3, \ldots \quad (33)$$

and

$$D_0 \neq 0, \ \ m = 0, \ \ (\gamma_0 = 0 \text{ is not an eigenvalue}) \quad (34)$$

for determining the eigenvalues γ_m. An expanded form of this determinant for any k is given in the appendix. Equation (33) has an infinite number of roots

$$\gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_m < \ldots \quad (35)$$

and for each of these roots are corresponding values of A_{im} and B_{im} in the eigenfunction $X_{im}(x)$, $i = 1, 2, 3, \ldots, k$ which can be determined to within a multiple of an arbitrary constant, and due to the form of equation (27) and a normalization process, which is explained in equations (36)–(44), the solution is unaffected by the value of this constant and with complete generality, it can be replaced by the number 1 (one), and $X_{im}(x)$, $i = 1, 2, 3, \ldots, k$ is completely specified.

The functions $X_{im}(x)$ and $X_{in}(x)$, $x_i \leqslant x \leqslant x_{i+1}$, $i = 1, 2, 3, \ldots, k$, m, n, integer, are orthogonal with respect to the weighting function

$$W_i = \sqrt{(\rho_i C_{pi})} \quad (36)$$

in the domain $[x_1, x_{k+1}]$. The orthogonality of these functions is

$$\sum_{i=1}^{k} \rho_i C_{pi} \int_{x_i}^{x_{i+1}} x^c X_{im}(x) X_{in}(x)\,dx = \begin{Bmatrix} \text{const.}, & m = n \\ 0, & m \neq n \end{Bmatrix} \quad (37)$$

and can be shown by integration by parts and then using the internal and external boundary conditions.

The function $L_{ij}(x)$ satisfies Dirichlet's conditions and if $Q_i(x, t)$ and $G_i(x)$ also satisfy Dirichlet's conditions, then these functions can be expanded in an infinite series of the eigenfunctions

$$L_{ij}(x) = \sum_{m=1}^{\infty} l_{mj} X_{im}(x) \quad (38)$$

$$\frac{Q_i(x, t)}{\rho_i C_{pi}} = \sum_{m=1}^{\infty} q_m(t) X_{im}(x) \quad (39)$$

$$G_i(x) = \sum_{m=1}^{\infty} g_m X_{im}(x). \quad (40)$$

When equations (38)–(40) are multiplied by $\rho_i C_{pi} x^c X_{in}(x)$ and then integrated with respect to x from x_i to x_{i+1} and then summed for all the values of i, the following relations for l_{mj}, $q_m(t)$ and g_m are obtained:

$$l_{mj} = \frac{1}{N_m} \sum_{i=1}^{k} \rho_i C_{pi} \int_{x_i}^{x_{i+1}} x^c L_{ij}(x) X_{im}(x)\,dx, \quad m = 1, 2, 3, \ldots \quad (41)$$

$$q_m(t) = \frac{1}{N_m} \sum_{i=1}^{k} \int_{x_i}^{x_{i+1}} x^c Q_i(x, t) X_{im}(x)\,dx, \quad m = 1, 2, 3, \ldots \quad (42)$$

$$g_m = \frac{1}{N_m} \sum_{i=1}^{k} \rho_i C_{pi} \int_{x_i}^{x_{i+1}} x^c G_i(x) X_{im}(x)\,dx, \quad m = 1, 2, 3, \ldots \quad (43)$$

where

$$N_m = \sum_{i=1}^{k} \rho_i C_{pi} \int_{x_i}^{x_{i+1}} x^c [X_{im}(x)]^2\,dx, \quad m = 1, 2, 3, \ldots \quad (44)$$

Substitution of equations (27), (38) and (39) into equation (7) gives

$$\sum_{m=1}^{\infty}\left[\frac{\mathrm{d}u_m(t)}{\mathrm{d}t}+\gamma_m^2 u_m(t)+\sum_{j=1}^{2} l_{mj}\frac{\mathrm{d}F_j(t)}{\mathrm{d}t}-q_m(t)\right]\times X_{im}(x)=0 \qquad (45)$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i=1,2,3,\ldots,k, \quad t \geqslant 0.$$

Equation (45) is satisfied if

$$\frac{\mathrm{d}u_m(t)}{\mathrm{d}t}+\gamma_m^2 u_m(t)=q_m(t)-\sum_{j=1}^{2} l_{mj}\frac{\mathrm{d}F_j(t)}{\mathrm{d}t}. \qquad (46)$$

The initial condition for equation (46) is obtained from equations (10), (27) and (40):

$$U_i(x,0)=\sum_{m=1}^{\infty} X_{im}(x)\,u_m(0)=G_i(x)=\sum_{m=1}^{\infty} g_m X_{im}(x) \qquad (47)$$

or

$$u_m(0)=g_m=v_m-\sum_{j=1}^{2} l_{mj}F_j(0), \quad m=1,2,3,\ldots \qquad (48)$$

where

$$v_m=\frac{1}{N_m}\sum_{i=1}^{k}\rho_i C_{pi}\int_{x_i}^{x_{i+1}} x^c V_i(x)\,X_{im}(x)\,\mathrm{d}x, \quad m=1,2,3,\ldots \qquad (49)$$

The solution of equation (46) is

$$u_m(t)=g_m\,\mathrm{e}^{-\gamma_m^2 t}+q_m(t)*\mathrm{e}^{-\gamma_m^2 t}-\sum_{j=1}^{2} l_{mj}\frac{\mathrm{d}F_j(t)}{\mathrm{d}t}*\mathrm{e}^{-\gamma_m^2 t}, \quad m=1,2,3,\ldots \qquad (50)$$

where the symbol * denotes convolution.

From equations (27) and (50), the solution for $U_i(x, t)$ can be written as

$$U_i(x,t)=\sum_{m=1}^{\infty}\left\{g_m\,\mathrm{e}^{-\gamma_m^2 t}+q_m(t)*\mathrm{e}^{-\gamma_m^2 t}-\sum_{j=1}^{2} l_{mj}\frac{\mathrm{d}F_j(t)}{\mathrm{d}t}*\mathrm{e}^{-\gamma_m^2 t}\right\}X_{im}(x) \qquad (51)$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i=1,2,3,\ldots,k, \quad t\geqslant 0$$

where

$$X_{im}(x)=A_{im}M_{im}(x)+B_{im}N_{im}(x)$$

and where $M_{im}(x)$ and $N_{im}(x)$ are given in Table 2. In equations (31) and (51), the following relations hold for A_{im} and B_{im}:

$$A_{1m}=1$$

and

$$B_{1m}=\frac{|b_1|}{|b|}, \quad A_{2m}=\frac{|b_2|}{|b|}, \quad B_{2m}=\frac{|b_3|}{|b|},\ldots$$

$$A_{im}=\frac{|b_{2i-2}|}{|b|}, \quad i=2,3,4,\ldots,k \qquad (53)$$

$$B_{im}=\frac{|b_{2i-1}|}{|b|}, \quad i=1,2,3,\ldots,k$$

where $|b|$ and $|b_\xi|$, $\xi = 1,2,3,\ldots,2k-1$ are obtained from the elements in the determinant D, equation (33), which is shown in the general expanded form in the Appendix. Using the elements in the determinant D, we may write

$$D=\begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1,2k}\\ b_{21} & b_{22} & \cdots & b_{2,2k}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ b_{2k-1,1} & b_{2k-1,2} & \cdots & b_{2k-1,2k}\\ b_{2k,1} & b_{2k,2} & & b_{2k,2k}\end{vmatrix}=0 \qquad (54)$$

and

$$|b|=\begin{vmatrix} b_{12} & b_{13} & \cdots & b_{1,2k}\\ b_{22} & b_{23} & \cdots & b_{2,2k}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ b_{2k-1,2} & b_{2k-1,3} & \cdots & b_{2k-1,2k}\end{vmatrix}\neq 0 \qquad (55)$$

which is of order $2k-1$. The determinant $|b_\xi|$, $\xi = 1, 2, 3, \ldots, 2k-1$, is formed by replacing the ξth column in $|b|$ by the column $(-b_{11} - b_{21} \ldots -b_{2k-1,1})$ formed from the corresponding elements in D.

Since all the components of $U_i(x, t)$ and $L_{ij}(x)$ are known, the temperature in each section is given by

$$T_i(x, t) = U_i(x, t) + \sum_{j=1}^{2} L_{ij}(x) F_j(t) \qquad (4)$$

$$x_i \leqslant x \leqslant x_{i+1}, \quad i = 1, 2, 3, \ldots, k, \quad t \geqslant 0$$

The solution $T_i(x, t)$ gives the temperature in any of the k plates, cylinders or spheres that are joined at their $k-1$ interfaces. The media have an arbitrary initial temperature distribution and are subject to heating or cooling at their mutual external boundaries from two different time dependent surroundings which exchange heat through different, though constant, film coefficients; in addition, the media is also subject to heating or cooling at each of the $k-1$ interfaces which also have arbitrary, different, though constant, film coefficients. For infinite values of the film coefficients at the interface, the problem will reduce to that of equal temperature at the interface. That is, as

$$h_\xi \to \infty, \quad \xi = 2, 3, 4, \ldots, k$$

the interfacial condition at $x = x_\xi$ will be

$$T_{\xi-1}(x_\xi, t) = T_\xi(x_\xi, t).$$

Thus at the interfacial boundaries, the present solution is able to handle either heating or cooling, temperature continuity or temperature discontinuities.

Numerical example

Consider the case of a three-layered composite of lead, aluminum and tin whose initial temperature is 0°C, Fig. 1. For time $t > 0$, the lead face at $x = x_1 = 0$ is raised to a temperature of 400°C while the rest of the composite is held at 0°C. The following property values have been used in the calculations:

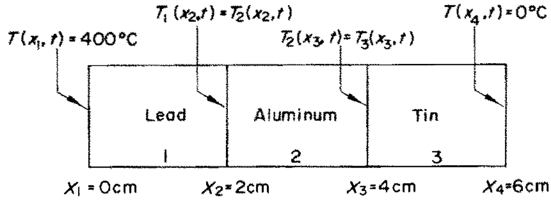

FIG. 1. Three layered composite.

$$x_1 = 0, \quad x_2 = 2\,\text{cm}, \quad x_3 = 4\,\text{cm}, \quad x_4 = 6\,\text{cm}$$

$$\rho_1 = 11{\cdot}08\,\text{g/cm}^3, \quad \rho_2 = 2{\cdot}71\,\text{g/cm}^3,$$

$$\rho_3 = 7{\cdot}4\,\text{g/cm}^3$$

$$C_{p1} = 0{\cdot}031\,\text{cal/g °C}, \quad C_{p2} = 0{\cdot}181\,\text{cal/g °C}$$

$$C_{p3} = 0{\cdot}054\,\text{cal/g °C}$$

$$K_1 = 297{\cdot}64\,\text{cal/cm °C h},$$

$$K_2 = 1741{\cdot}18\,\text{cal/cm °C h},$$

$$K_3 = 565{\cdot}51\,\text{cal/cm °C/h}.$$

The transcendental equation for determining the eigenvalues is given by equation (56). This equation (56) has an infinite number of roots γ_m and for each of these roots there are corresponding values of A_{im} and B_{im} in the eigenfunction $X_{im}(x)$. Table 3 lists the first 20 eigenvalues for this particular problem.

Table 3.

m	γ_m	m	γ_m
1	16·6565	11	227·7768
2	44·20136	12	239·4976
3	58·3468	13	267·4471
4	83·6840	14	283·3284
5	99·9762	15	303·6040
6	122·2505	16	324·3276
7	139·9455	17	347·2548
8	165·6957	18	366·0599
9	182·8667	19	383·1424
10	199·6942	20	410·3552

When the eigenvalues for the problem are determined the solution for the temperature distribution in each region of the composite

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \cos\frac{\gamma_m}{D_1}x_2 & \sin\frac{\gamma_m}{D_1}x_2 & -\cos\frac{\gamma_m}{D_2}x_2 & -\sin\frac{\gamma_m}{D_2}x_2 & 0 & 0 \\ \frac{K_1 D_2}{K_2 D_1}\sin\frac{\gamma_m}{D_1}x_2 & \frac{K_1 D_2}{K_2 D_1}\cos\frac{\gamma_m}{D_1}x_2 & \sin\frac{\gamma_m}{D_2}x_2 & -\cos\frac{\gamma_m}{D_2}x_2 & 0 & 0 \\ 0 & 0 & \cos\frac{\gamma_m}{D_2}x_3 & \sin\frac{\gamma_m}{D_2}x_3 & -\cos\frac{\gamma_m}{D_3}x_3 & -\sin\frac{\gamma_m}{D_3}x_3 \\ 0 & 0 & -\frac{K_2}{K_3}\frac{D_3}{D_2}\sin\frac{\gamma_m}{D_2}x_3 & \frac{K_2}{K_3}\frac{D_3}{D_2}\cos\frac{\gamma_m}{D_2}x_3 & \sin\frac{\gamma_m}{D_3}x_3 & -\cos\frac{\gamma_m}{D_3}x_3 \\ 0 & 0 & 0 & 0 & \cos\frac{\gamma_m}{D_3}x_4 & \sin\frac{\gamma_m}{D_3}x_4 \end{vmatrix} = 0 \qquad (56)$$

can be written by using the methods outlined in equations (17)–(22) and equations (38)–(51). The results are

$$T_1(x,t) = 400(A_{11}x + B_{11}) + \sum_{m=1}^{\infty} g_m\, e^{(-\gamma_m^2 t)} \sin\frac{\gamma_m x}{D_1} \quad (57)$$

$$T_2(x,t) = 400(A_{21}x + B_{21}) + \sum_{m=1}^{\infty} g_m\, e^{(-\gamma_m^2 t)} X_{2m}(x) \quad (58)$$

$$T_3(x,t) = 400(A_{31}x + B_{31}) + \sum_{m=1}^{\infty} g_m\, e^{(-\gamma_m^2 t)} X_{3m}(x) \quad (59)$$

where

$$K_{EQ} = K_1K_3(x_3 - x_2) + K_1K_2(x_4 - x_3) + K_2K_3(x_2 - x_1)$$

$$A_{11} = -\frac{K_2K_3}{K_{EQ}}, \quad A_{21} = -\frac{K_1K_3}{K_{EQ}},$$

$$A_{31} = -\frac{K_1K_2}{K_{EQ}}$$

$$B_{11} = 1{\cdot}0, \quad B_{21} = \frac{K_1K_3x_3 + K_1K_2(x_4 - x_3)}{K_{EQ}},$$

$$B_{31} = \frac{K_1K_2x_4}{K_{EQ}}$$

$$N_m = \frac{\rho_1C_{p1}x_2}{2} + \frac{\rho_2C_{p2}(x_3 - x_2)}{2} \times \left[\sin^2\frac{\gamma_m x_2}{D_1} + \left(\frac{K_1D_2}{K_2D_1}\right)^2 \cos^2\frac{\gamma_m(x_2)}{D_1}\right] + \frac{\rho_cC_{p3}(x_4 - x_3)}{2} \left[\frac{\frac{K_1D_2}{K_2D_1}\cos\frac{\gamma_m x_2}{D_1}\sin\frac{\gamma_m(x_3 - x_2)}{D_2}}{\sin\frac{\gamma_m(x_4 - x_3)}{D_3}} + \frac{\sin\frac{\gamma_m x_2}{D_1}\cos\frac{\gamma_m(x_3 - x_2)}{D_2}^{\,2}}{\sin\frac{\gamma_m(x_4 - x_3)}{D_3}}\right]^2$$

$$g = \frac{-400\,\rho_1C_{p1}D_1}{\gamma_m N_m}.$$

$$X_{2m}(x) = \left[\frac{K_1D_2}{K_2D_1}\cos\frac{\gamma_m}{D_1}x_2\right]\sin\frac{\gamma_m(x - x_2)}{D_2} \left[\sin\frac{\gamma_m}{D_1}x_2\right]\cos\frac{\gamma_m(x' - x_2)}{D_2}$$

$$X_{3m}(x) = -\left[\frac{K_1D_2}{K_2D_1}\cos\frac{\gamma_m x_2}{D_1}\sin\frac{\gamma_m(x_3 - x_2)}{D_2} + \sin\frac{\gamma_m x_2}{D_1}\cos\frac{\gamma_m(x_3 - x_2)}{D_2}\right]\frac{\sin\frac{\gamma_m(x - x_4)}{D_3}}{\sin\frac{\gamma_m(x_4 - x_3)}{D_3}}.$$

The resulting temperature distribution is given in Figs. 2 and 3 for various values of position x and time t.

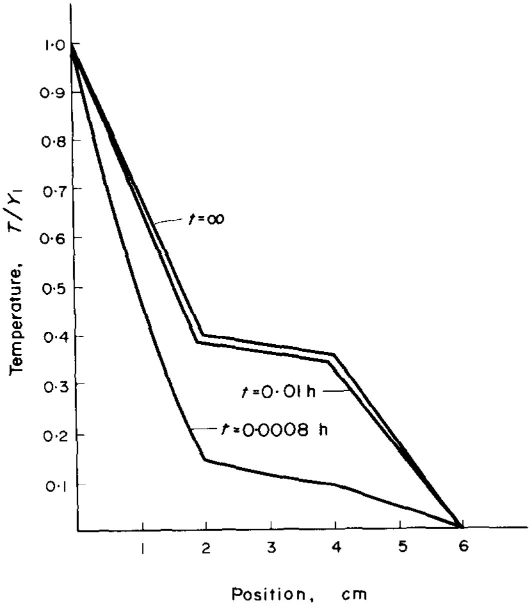

FIG. 2. Temperature distribution.

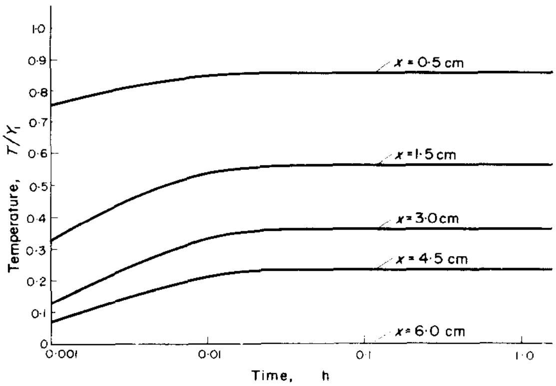

FIG. 3. Temperature–time history within composite.

REFERENCES

1. H. L. BEACH, The application of the orthogonal expansion technique to conduction heat transfer problems in multilayer cylinders, M.S. Thesis, Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, North Carolina, (1967).
2. P. E. BULAVIN and V. M. KASCHEEV, Solution of the non-homogeneous heat conduction equation for multilayered bodies, *Int. Chem. Engng* **5**, 112–115 (1965).
3. H. S. CARSLAW and J. C. JAEGER, *Conduction of Heat in Solids*, 2nd Edn, pp. 305–347. Clarendon Press, Oxford (1959).
4. J. CRANK, *Mathematics of Diffusion*. Clarendon Press, Oxford (1965).
5. C. J. MOORE, Heat transfer across surfaces in contact: studies of transients in one-dimensional composite systems, Ph.D. Thesis, Mechanical Engineering Department, Southern Methodist University, Dallas, Texas (1967).
6. M. H. OZISIK, *Boundary Value Problems of Heat Conduction*, pp. 273–298. International Textbook Co., Scranton, Pa. (1968).
7. C. W. TITTLE, Boundary value problems in composite media, *J. Appl. Phys.* **36**, 1486–1488 (1965).
8. V. VODICKA, Wärmeleitung in geschichteten Kugel- und Zylinderkörpern, *Schweizer Archiv* **10**, (1950).
9. V. VODICKA, Eindimensionale Wärmeleitung in geschichteten Körpern, *Mathematische Nachrichten* **14** (1955).

APPENDIX

The expanded form of the determinant, equation (33), that gives the transcendental equation for determining the eigenvalues γ_m is given in equation (A.8). In equation (A.8), the following identities hold:

$$M^*_{1m}(x_1) = M'_{1m}(x_1) - \frac{h_1}{K_1} M_{1m}(x_1) \tag{A.1}$$

$$N^*_{1m}(x_1) = N'_{1m}(x_1) - \frac{h_1}{K_1} N_{1m}(x_1) \tag{A.2}$$

$$D = \begin{vmatrix}
M^*_{1m}(x_1) & N^*_{1m}(x_1) & 0 & 0 & 0 & 0 & 0 \\
M^*_{2m}(x_2) & N^*_{2m}(x_2) & -M_{2m}(x_2) & -N_{2m}(x_2) & 0 & 0 & 0 \\
\lambda_1 M'_{im}(x_2) & \lambda_1 N'_{im}(x_2) & -M'_{2m}(x_2) & -N'_{2m}(x_2) & 0 & 0 & 0 \\
0 & 0 & M^*_{3m}(x_3) & N^*_{3m}(x_3) & -M_{3m}(x_3) & -N_{3m}(x_3) & 0 \\
0 & 0 & \lambda_2 M'_{2m}(x_3) & \lambda_2 N'_{2m}(x_3) & -M'_{3m}(x_3) & -N'_{3m}(x_3) & 0 \\
0 & 0 & 0 & 0 & M^*_{4m}(x_4) & N^*_{4m}(x_4) & -M_{4m}(x \\
0 & 0 & 0 & 0 & \lambda_3 M_{3m}(x_4) & \lambda_3 N'_{3m}(x_4) & -M'_{4m}(x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}$$

$$M^*_{im}(x_i) = \frac{K_{i-1}}{h_i} M'_{i-1,m}(x_i) + M_{i-1,m}(x_i),$$

$$i = 2, 3, 4, \ldots, k \qquad \text{(A.3)}$$

$$N^*_{im}(x_i) = \frac{K_{i-1}}{h_i} N'_{i-1,m}(x_i) + N_{i-1,m}(x_i),$$

$$i = 2, 3, 4, \ldots, k \qquad \text{(A.4)}$$

$$M^*_{k+1,m}(x_{k+1}) = M'_{km}(x_{k+1}) + \frac{h_{k+1}}{K_k} M_{km}(x_{k+1}) \qquad \text{(A.5)}$$

$$N^*_{k+1,m}(x_{k+1}) = N'_{km}(x_{k+1}) + \frac{h_{k+1}}{K_k} N_{km}(x_{k+1}) \qquad \text{(A.6)}$$

and

$$\lambda_i = K_i/K_{i+1}, \quad i = 1, 2, \ldots, k-1 \qquad \text{(A.7)}$$

where the $M_{im}(x)$ and $N_{im}(x)$ are given in Table 2.

$$\left.\begin{array}{ccccccc}
0 & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
0 & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
0 & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
0 & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
0 & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
-N_{4m}(x_4) & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
-N'_{4m}(x_4) & \cdot\ \cdot\ \cdot & 0 & 0 & 0 & 0 \\
\cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot\ \cdot\ \cdot & M^*_{km}(x_k) & N^*_{km}(x_k) & -M_{km}(x_k) & -N_{km}(x_k) \\
0 & \cdot\ \cdot\ \cdot & \lambda_{k-1}M'_{k-1}(x_k) & \lambda_{k-1}N'_{k-1}(x_k) & -M'_{km}(x_k) & -N'_{km}(x_k) \\
0 & \cdot\ \cdot\ \cdot & 0 & 0 & M^*_{k+1}(x_{k+1}) & N^*_{k+1}(x_{k+1})
\end{array}\right] = 0 \qquad \text{(A.8)}$$

DIFFUSION A TRAVERS DES MILIEUX COMPOSITES

Résumé—On a résolu l'équation de diffusion pour une distribution de température dans chacune des k sections d'un composite avec génération de chaleur interne et aux $k - 1$ interfaces, soit un chauffage, soit un refroidissement, soit un contact thermique parfait. Le composite consiste en k plaques, cylindres ou sphères discrets, chacun étant de matière différente. Les k sections ont une distribution de température initiale arbitraire et les conditions d'échanges de chaleur aux frontières externes s'exprimant par deux coefficients de film différents, constants et arbitraires avec deux environnements différents dépendants du temps.

On recherche la solution en utilisant une substitution à variable unique dépendante qui donne une nouvelle équation aux dérivées partielles avec des conditions limites homogènes externes. En utilisant la relation d'orthogonalité de Vodicka, on obtient la solution de cette équation aux dérivées partielles.

Cette solution donne la distribution de température dans chaque élément pour une position X quelconque en fonction du temps t et pour le type le plus général de conditions initiales et aux limites.

DIFFUSION DURCH ZUSAMMENGESETZTE MEDIEN

Zusammenfassung—Es wird die Diffusionsgleichung in jedem der k Abschnitte eines Verbundsystems mit inneren Wärmequellen gelöst: an den $k - 1$ Zwischenflächen kann dabei Heizung. Kühlung oder vollkommener thermischer Kontakt vorgesehen sein. Der Verbundkörper besteht aus k Platten. Zylinder- oder Kugelschalen aus unterschiedlichem Material.

Die k Abschnitte haben eine beliebige Anfangstemperaturverteilung. und die Medien tauschen an den äusseren Grenzflächen Wärme durch zwei verschiedene. aber konstante. beliebig grosse Wärmeübergangskoeffizienten mit zwei verschiedenen. zeitabhängigen Umgebungen aus.

Die Lösung wird erhalten durch Verwendung einer eindeutigen Substitution für die abhängige Variable. welche eine neue partielle Differentialgleichung mit homogenen äusseren Randbedingungen liefert. Die Lösung für diese abgeleitete partielle Differentialgleichung wird dann unter Verwendung einer Orthogonalitätsrelation nach Vodicka gefunden. Die so erhaltene Lösung liefert die Temperaturverteilung in allen k Platten. Zylinder- oder Kugelschalen an jeder Stelle x und zu jeder Zeit t für den allgemeinsten Typ linearer Rand-. Innen- und Anfangsbedingungen.

ДИФФУЗИЯ ЧЕРЕЗ СОСТАВНЫЕ СРЕДЫ

Аннотация—Решено уравнение диффузии для распределения температуры в каждой из к секций составного тела с внутренними источниками тепла и нагревом, охлаждением или идеальным термическим контактом на к–I поверхностях раздела. Составное тело состоит из к дискретных пластин, цилиндров или сфер, изготовленных из различных материалов.

В к секциях задано произвольное начальное распределение температуры, а теплообмен на внешних границах задан различными, зависящими от времени граничными условиями.

Решение получено с помощью универсальной подстановки, которая приводит к дифференциальному уравнению в частных производ—с однородными граничными условиями. Решение получается с помощью соотношения ортогональности типа соотношений Водички.

Полученное таким образом решение дает распределение температуры в любой из к пластин, цилиндров или сфер при любом положении x и времени для наиболее общего типа линейных граничных, внутренних и начальных условий.